

POWERS OF GENERIC IDEALS AND THE WEAK LEFSCHETZ PROPERTY FOR POWERS OF SOME MONOMIAL COMPLETE INTERSECTIONS

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ABSTRACT. Given an ideal $I = (f_1, \dots, f_r)$ in $\mathbb{C}[x_1, \dots, x_n]$ generated by forms of degree d , and an integer $k > 1$, how large can the ideal I^k be, i.e., how small can the Hilbert function of $\mathbb{C}[x_1, \dots, x_n]/I^k$ be? If $r \leq n$ the smallest Hilbert function is achieved by any complete intersection, but for $r > n$, the question is in general very hard to answer. We study the problem for $r = n + 1$, where the result is known for $k = 1$. We also study a closely related problem, the Weak Lefschetz property, for S/I^k , where I is the ideal generated by the d 'th powers of the variables.

1. INTRODUCTION

Let $I = (f_1, \dots, f_r)$ be an ideal generated by forms of degree d in $\mathbb{C}[x_1, \dots, x_n]$, and let $k \geq 1$. How large can the ideal I^k be, i.e., how small can the Hilbert function of $\mathbb{C}[x_1, \dots, x_n]/I^k$ be? It is clear that we get the smallest Hilbert series if the f_i 's are general. If $r \leq n$, I is a complete intersection, and the Hilbert series (and even the graded Betti numbers) for I^k are known ([5]). But if $r > n$ not much is known even if $k = 1$, and we are not aware of any result for $k > 1$. For $k = 1$ the main classes for which the result is known is when $n \leq 3$ ([1],[4]) or when $r = n + 1$ ([8]). In all known cases the series for an ideal with r general generators of degree $d \geq 2$ in n variables is $[(1 - t^d)^r / (1 - t)^n]$, where $[\sum_{i \geq 0} a_i t^i]$ means truncate before the first nonpositive term. A first guess for the Hilbert series of $\mathbb{C}[x_1, \dots, x_n]/I^k$, I generated by r general forms of degree d , could be that the series for $k \gg 1$ and $r > n$ is the same as for $\binom{k+r-1}{r-1}$ general forms of degree dk . We will show that this is not always the case for geometric reasons.

We also study a closely related problem, the *Weak Lefschetz property* (WLP) for powers of some monomial complete intersections ideals. Recall that a graded algebra A satisfies the WLP if there exists a linear form L such that the multiplication map $\times L: A_i \rightarrow A_{i+1}$ has maximal rank for all degrees i . See [7] for a survey on the WLP.

For a graded algebra $R = \oplus R_i$ we let $R(t)$ denote the Hilbert series of R .

2. POWERS OF IDEALS OF GENERAL FORMS

Let f_1, f_2, \dots, f_{n+1} be general forms of degree d in $\mathbb{C}[x_1, \dots, x_n]$. We will start by showing that when $k = d^{n-1}$, the dimension of $\mathbb{C}[x_1, \dots, x_n]/(f_1, f_2, \dots, f_{n+1})^k$ in degree dk is one less than expected.

Lemma 1. *For general forms f_1, f_2, \dots, f_{n+1} of degree d in $\mathbb{C}[x_1, \dots, x_n]$ there is exactly one relation of degree d^{n-1} in $\mathbb{C}[f_1, \dots, f_{n+1}]$.*

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Proof. General forms f_1, f_2, \dots, f_{n+1} give a well-defined map $\Phi: \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n$ since they have no common zeroes. The closure of the image is a variety of dimension $n-1$, so it is a hypersurface. The inverse image of a general line gives a reduced complete intersection, as we can see from producing one such example by choosing the forms to be powers of general linear forms and by considering a line that gives the inverse image given by the radical complete intersection ideal $(f_1 - f_n, f_2 - f_n, \dots, f_{n-1} - f_n)$. Thus a general line meets the hypersurface in d^{n-1} distinct points and hence the image is not contained in any hypersurface of lower degree than d^{n-1} . We conclude that there is a unique equation of degree d^{n-1} defining the image of Φ . \square

In the next two subsections, we will focus on the situation in two and three variables. We will need the following lemma.

Lemma 2. *Let I be an ideal in $\mathbb{C}[x_1, \dots, x_n]$ generated by homogeneous elements in degree d and suppose that $x_i^d \in I$ for $i = 1, \dots, n$. Suppose that $\mathbb{C}[x_1, \dots, x_n]/I^k$ is zero in degree a and that $a \geq d(n-1)$. Then $\mathbb{C}[x_1, \dots, x_n]/I^{k+1}$ is zero in degree $a+d$.*

Proof. Let $R = \mathbb{C}[x_1, \dots, x_n]$ and write $R = R_0 \oplus R_1 \oplus \dots$. By the assumption, $R_a \subset I^k$, so $R_a x_i^d \subset I^{k+1}$ for $i = 1, \dots, n$. Let m be a monomial in R_{a+d} . Since $a+d \geq dn$, there is at least one variable, say x_j , whose exponent in m is greater than or equal to d . But then $m \in R_a x_j^d$, that is, $m \in I^{k+1}$. \square

2.1. The case $n = 2, r = 3$. Let $R_{2,d,k} = \mathbb{C}[x, y]/(f_1, f_2, f_3)^k$, where the f_i 's are general forms of degree d . If $d = 2$, then $(f_1, f_2, f_3) = (x, y)^2$, so $R_{2,2,k} = \mathbb{C}[x, y]/(x, y)^{2k}$. We assume in this section that $d \geq 3$.

Lemma 3. *If $k \geq d-2$, then $R_{2,d,k}(t) \geq \sum_{i=0}^{dk-1} (i+1)t^i + \binom{d-1}{2} t^{dk}$ coefficientwise.*

Proof. Consider the case when $k = d$. By Lemma 1 the dimension of $R_{2,d,k}$ in degree dk is $d^2 + 1 - \left(\binom{d+2}{2} - 1 \right) = \binom{d-1}{2}$. This implies that in degree $k \geq d$, there are $\binom{k-d+2}{2}$ relations among the monomials of degree k in the f_i 's. We have $\binom{k+2}{2}$ monomials of degree k in the f_i 's. Thus the dimension of $R_{2,d,k}$ in degree dk is at least $dk + 1 - \left(\binom{k+2}{2} - \binom{k-d+2}{2} \right) = \binom{d-1}{2}$.

Finally, let us consider the case when $k = d-1$ and $k = d-2$. The dimension of $R_{2,d,k}$ in degree dk is at least $dk + 1 - \binom{k+2}{2}$. If $k = d-1$ or $k = d-2$, this equals $\binom{d-1}{2}$. \square

Lemma 4. *If $k \geq d-2$, the Hilbert function of $\mathbb{C}[x, y]/(x^d, y^d, x^{d-1}y)^k$ in degree dk is $\binom{d-1}{2}$.*

Proof. By Lemma 3, we have $\mathbb{C}[x, y]/(x^d, y^d, x^{d-1}y)^k(t) \geq \sum_{i=0}^{dk-1} (i+1)t^i + \binom{d-1}{2} t^{dk}$. Thus it is enough to show that the Hilbert function of $\mathbb{C}[x, y]/(x^d, y^d, x^{d-1}y)^k$ in degree dk is at most $\binom{d-1}{2}$.

Let $h = \min(k, d-1)$. We construct $h+1$ separate groups indexed by $0, \dots, h$, where we let the b 'th group consist of the polynomials $x^{da}(xy^{d-1})^b y^{dc}$ of degree dk such that $a+b \leq k$. The leading monomials in this group are

$$x^b y^{dk-b}, x^{b+d} y^{dk-(b+d)}, \dots, x^{b+(k-b)d} y^{dk-(b+(k-b)d)},$$

so there are $k - b + 1$ leading monomials in group b .

In total this gives $\sum_{b=0}^h (k - b + 1) = (h + 1)(k + 1) - \binom{h+1}{2}$ elements. If $k = d - 2$, then the dimension in degree dk is at most $d(d - 2) + 1 - (d - 1)(d - 1) + \binom{d-1}{2} = \binom{d-1}{2}$. If $k = d - 1$, then the dimension in degree dk is at most $d(d - 1) + 1 - d^2 + \binom{d}{2} = \binom{d-1}{2}$. Finally, if $k \geq d$, the dimension in degree dk is at most $kd + 1 - d(k + 1) + \binom{d}{2} = \binom{d-1}{2}$. \square

Lemma 5. *The Hilbert function of the algebra $\mathbb{C}[x, y]/(x^d, y^d, x^{d-1}y + xy^{d-1})^k$ is zero in degree $dk + 1$ if $k \geq \max(d - 3, 1)$.*

Proof. We will prove by induction that all the monomials of degree $dk + 1$ are in the ideal $J = (x^d, y^d, x^{d-1}y + xy^{d-1})^k$. Assume by induction on ℓ that we have all the monomials of the form $x^{id+j}y^{(k-i)d+1-j}$ and $x^{(k-i)d+1-j}y^{id+j}$ in J for $0 \leq i \leq k - j$, $0 \leq j \leq \ell + 1$. Denote the ideal generated by these monomials by $J^{(\ell)}$. Observe that $J^{(0)} = (x, y)(x^d, y^d)^k \subseteq J$ by the definition of J .

We compute for $0 \leq i \leq k - \ell - 1$ the following element in J

$$\begin{aligned} x \cdot (x^{d-1}y + xy^{d-1})^{\ell+1} \cdot x^{id}y^{(k-i-\ell-1)d} &= \sum_{j=0}^{\ell+1} \binom{\ell+1}{j} x^{(i+j)d+\ell+2-2j} y^{(k-i-j)d-\ell-1+2j} \\ &\equiv x^{id+\ell+2} y^{(k-i)d-\ell-1} \pmod{J^{(\ell)}} \end{aligned}$$

since all but the first term are in $J^{(\ell)}$. By symmetry we also get that the following is in J

$$\begin{aligned} y \cdot (x^{d-1}y + xy^{d-1})^{\ell+1} \cdot x^{(k-i-\ell-1)d} y^{id} &= \sum_{j=0}^{\ell+1} \binom{\ell+1}{j} x^{(k-i-j)d-\ell-1+2j} y^{(i+j)d+\ell+2-2j} \\ &\equiv x^{(k-i)d-\ell-1} y^{id+\ell+2} \pmod{J^{(\ell)}} \end{aligned}$$

since all but the last term are in $J^{(\ell)}$. Hence we have proved that $J^{(\ell+1)} \subseteq J$ for $0 \leq \ell \leq k$.

Now we need to prove that $J^{(k)}$ contains all monomials of degree $dk + 1$. Any monomial of degree $dk + 1$ can be written as $x^{jd+r}y^{(k-j)d-(r-1)}$, where $0 \leq r < d$. This monomial is of the form $x^{id+\ell+2}y^{(k-i)d-\ell-1}$ in $J^{(k)}$ if $r = \ell + 2$ and $j \leq k - \ell - 1 = k - r + 1$, i.e., $r \leq k - j + 1$ and it is of the form $x^{(k-i)d-\ell-1}y^{id+\ell+2}$ in $J^{(k)}$ if $jd + r = (k - i)d - \ell - 1$, i.e., $j = k - i - 1$, $r = d - \ell - 1$ and $0 \leq i \leq k - \ell - 1$, i.e., $0 \leq k - j - 1 \leq r + k - d$, which can be written as $r \geq d - j - 1 \geq d - k$. We will get all monomials in degree $dk + 1$ if $d - j - 1 \leq (k - j + 1) + 1$, i.e., when $d - 3 \leq k$. \square

Theorem 6. *If $k \geq d - 2$, then*

$$R_{2,d,k}(t) = \sum_{i=0}^{dk-1} (i+1)t^i + \binom{d-1}{2} t^{dk}.$$

The Betti numbers of $R_{2,d,k}$ are $\beta_{0,0} = 1$, $\beta_{1,dk} = \frac{2dk-d^2+3d}{2}$, $\beta_{2,dk+1} = (dk - d^2 + 3d - 2)$, $\beta_{2,dk+2} = \binom{d-1}{2}$.

Proof. By Lemma 4 and Lemma 5, $R_{2,d,k}(t) \leq \sum_{i=0}^{dk-1} (i+1)t^i + \binom{d-1}{2}t^{dk}$. Together with Lemma 3 we get the claimed Hilbert series.

The Hilbert series equals $(\sum_{i=0}^{dk+2} (-1)^i \beta_{i,j} t^{i+j}) / (1-t)^2$. We know that the number of generators of $(f_1, f_2, f_3)^k$ are $\binom{k+2}{2} - \binom{k-d+2}{2} = \frac{2dk-d^2+3d}{2}$, and they are all of degree dk , so $\beta_1 = \beta_{1,dk} = \frac{2dk-d^2+3d}{2}$. Also $\beta_{2,dk+2} = \binom{d-1}{2}$ and $\beta_{2,i} = 0$ if $i > dk+2$ since $R_{2,d,k}$ has socle of dimension $\binom{d-1}{2}$ in degree dk , and no socle in higher degree. \square

Remark 7. We are convinced that also the algebra $\mathbb{C}[x, y] / (x^d, y^d, (x+y)^d)^k$, is zero in degree $dk+1$ for any d and $k \geq \max(d-3, 1)$, but we have not been able to prove this. See also Conjecture 1.

2.2. The case $n = 3$, $r = 4$.

Theorem 8. *The Hilbert series of $R_{3,2,k}$ is*

$$\sum_{i=0}^{2k-1} \binom{i+2}{2} t^i + (3k-1)t^{2k}.$$

Proof. A similar calculation as in the proof of Lemma 3 shows that $R_{3,2,k}$ has at least dimension $\binom{2k+2}{2} - \binom{k+3}{3} + \binom{k-4+3}{3} = 3k-1$ in degree $2k$. Thus we have an inequality. That there is equality follows from the following lemma. \square

Lemma 9. *The Hilbert series of*

$$\mathbb{C}[x, y, z] / (x^2, y^2, z^2, (x+y+z)^2)^k = \mathbb{C}[x, y, z] / (x^2, y^2, z^2, xy+xz+yz)^k$$

is

$$\sum_{i=0}^{2k-1} \binom{i+2}{2} t^i + (3k-1)t^{2k}.$$

Proof. It is enough to show that the Hilbert series has at most dimension $3k-1$ in degree $2k$ and that the algebra is zero in degree $2k+1$.

We first consider degree $2k$. We claim that all monomials of degree $2k$ except the $3k-1$ monomials

$$x^{2k-1}z, x^{2k-3}z^3, \dots, xz^{2k-1}, y^{2k-1}z, y^{2k-3}z^3, \dots, yz^{2k-1}, xy^{2k-2}z, xy^{2k-4}z^3, \dots, xy^2z^{2k-3}$$

occur as leading monomials in lexicographic ordering. We have $(x^2, y^2, z^2, (x+y+z)^2) = (x^2, y^2, z^2, f)$, where $f = xy + xz + yz$. If $m = x^a y^b z^c$ is a monomial of degree $2k$, we let $t(m) = (a \pmod{2}, b \pmod{2}, c \pmod{2})$. If $t(m) = (0, 0, 0)$, then m lies in the ideal. If $t(m) = (1, 1, 0)$, then $m = (xy)^{\min(a,b)} M$, where $t(M) = (0, 0, 0)$, so m is the leading monomial of $f^{\min(a,b)} M$. If $t(m) = (1, 0, 1)$ we claim that m is a leading monomial except when $b = 0$ or $a = 1$. If $t(m) = (0, 1, 1)$ we claim that m is a leading monomial except when $a = 0$. To prove the claims, it suffices to show that all monomials of the form $x^2 y z M$, $t(M) = (0, 0, 0)$ and all monomials of the form $x^3 y^2 z M$, $t(M) = (0, 0, 0)$ are leading. Now $x^2 y z$ is the leading monomial of $f^2 - x^2 y^2$, and $x^3 y^2 z$ is the leading monomial of $f(f^2 - x^2 y^2 - x^2 z^2)$.

We now consider degree $2k+1$. A calculation shows that $\mathbb{C}[x, y, z] / (x^2, y^2, z^2, xy + xz + yz)$ is zero in degree three and that $\mathbb{C}[x, y, z] / (x^2, y^2, z^2, xy + xz + yz)^2$ is zero in

degree five. Since $2k+1 \geq 2(n-1)$ when $k \geq 2$, the remaining cases follow from Lemma 2. \square

Theorem 10. *The Hilbert series of $R_{3,3,k}$ is $\sum_{i=0}^{3k-1} \binom{i+2}{2} t^i + (27k-56)t^{3k}$ when $9 \leq k \leq 40$ and $[(1-t^{3k})\binom{k+3}{3}]/(1-t)^3$ when $k < 9$.*

Proof. Consider first the case $k \geq 9$. By Lemma 1 and a similar calculation as in the proof of Lemma 3, we get that $R_{3,3,k}$ has at least dimension $\binom{3k+2}{2} - \binom{k+3}{3} + \binom{k-6}{3} = 27k-56$ in degree $3k$.

Next, consider the case $k < 9$. If the $\binom{k+3}{3}$ generators were generic, the series would be $[(1-t^{3k})\binom{k+3}{3}]/(1-t)^3$, so the series is a lower bound.

To get an upper bound, it is enough to find an example. According to Macaulay2, the ideal $(x^3, y^3, z^3, x^2y+11xy^2-50x^2z+48xyz-29y^2z-9xz^2+30yz^2) \subset \mathbb{Z}/101\mathbb{Z}[x_1, x_2, x_3]$ has the desired property for $k \leq 40$. \square

Remark 11. We are convinced that $R_{3,3,k}(t) = \sum_{i=0}^{3k-1} \binom{i+2}{2} t^i + (27k-56)t^{3k}$ in general, cf. Conjecture 1.

2.3. The general case. We believe that some of the results have generalizations to any n . By Lemma 1, there is unique relation among the $\binom{d^{n-1}+n}{n}$ generators of I^k , that is, $\binom{d^n+n-1}{n-1} \geq \binom{d^{n-1}+n}{n} - 1$. To show that this relation is not trivial, we need to show that there is room for $\binom{d^{n-1}+n}{n}$ generators of degree d^n in $\mathbb{C}[x_1, \dots, x_n]$. When $(d, n) = (2, 2)$, we have five generators but only room for four. When $(d, n) = (3, 2)$, we have ten generators and room for ten. In all other cases, we have a strict inequality.

Lemma 12. *Let $d, n \geq 2$, $(d, n) \neq (2, 2), (3, 2)$. Then $\binom{d^n+n-1}{n-1} > \binom{d^{n-1}+n}{n}$.*

Proof. We need to show that

$$n(d^n + n - 1) \cdots (d^n + 1) > (d^{n-1} + n) \cdots (d^{n-1} + 1).$$

We have $(d^n + n - 1) \cdots (d^n + 1) > (d^n + 1)^{n-1}$ and $(d^{n-1} + n) \cdot (d^{n-1} + n - 1)^{n-1} > (d^{n-1} + n) \cdots (d^{n-1} + 1)$. Thus we are done if we can show that

$$n \cdot \left(\frac{d^n + 1}{d^{n-1} + n - 1} \right)^{n-1} > d^{n-1} + n.$$

Now $\frac{d^n+1}{d^{n-1}+n-1} = \frac{d \cdot (d^{n-1}+n-1)+1-dn+d}{d^{n-1}+n-1} > d(1 - \frac{n}{d^{n-1}})$. We have $(1 - \frac{n}{d^{n-1}})^{n-1} = \binom{n-1}{n-1} - \binom{n-1}{n-1-1} \frac{n}{d^{n-1}} + \binom{n-1}{n-1-2} \frac{n^2}{(d^{n-1})^2} - \binom{n-1}{n-1-3} \frac{n^3}{(d^{n-1})^3} + \cdots$ and $\binom{n-1}{n-1-a} \frac{n^a}{(d^{n-1})^a} - \binom{n-1}{n-1-(a+1)} \frac{n^{a+1}}{(d^{n-1})^{a+1}} = \binom{n-1}{n-1-a} \frac{n^a}{(d^{n-1})^a} \left(1 - \frac{n-a-1}{a+1} \cdot \frac{n}{d^{n-1}} \right) > \binom{n-1}{n-1-a} \frac{n^a}{(d^{n-1})^a} \left(1 - \frac{n^2}{d^{n-1}} \right)$.

The inequality $1 - \frac{n^2}{d^{n-1}} \geq 0 \Leftrightarrow d^{n-1} \geq n^2$ holds true for $d \geq 3, n \geq 3$ and for $d = 2, n \geq 7$. In these cases, we get the inequality $(1 - \frac{n}{d^{n-1}})^{n-1} > 1 - \frac{n^2}{d^{n-1}}$, so it is then enough to show that

$$nd^{n-1}(1 - \frac{n^2}{d^{n-1}}) > d^{n-1} + n.$$

We have $nd^{n-1}(1 - \frac{n^2}{d^{n-1}}) > d^{n-1} + n \Leftrightarrow d^{n-1} > \frac{n^3+n}{n-1}$. This inequality holds true for $d \geq 2, n \geq 7$ and $n \geq 3, d \geq 3$.

We are left with a few special cases only. In the $n = 2$ case, we have to check the inequality $2 \cdot \frac{d^2+1}{d+1} > d+2 \Leftrightarrow d^2 > 3d$, which holds when $d \geq 4$.

The remaining cases are covered by the conditions $3 \leq n \leq 6, d = 2$ and then the inequality $\binom{d^n+n-1}{n-1} > \binom{d^{n-1}+n}{n}$ is checked to be true by hand. \square

By Lemma 1 and 12, the dimension of $R_{n,d,k}$ in degree dk is at least $\binom{dk+n-1}{n-1} - \binom{k+n}{n} + \binom{k-d^{n-1}+n}{n}$ when $k \geq d^{n-1}$, with equality when $k = d^{n-1}$. We conjecture that the algebra is zero in higher degrees.

Conjecture 1. *If $k \geq d^{n-1}$ and $(d, n) \neq (2, 2)$, the Hilbert series of the algebra $\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_{n+1})^k$, f_i general of degree d , is*

$$\sum_{i=0}^{dk-1} \binom{i+n-1}{n-1} t^i + \left(\binom{dk+n-1}{n-1} - \binom{k+n}{n} + \binom{k-d^{n-1}+n}{n} \right) t^{dk}.$$

Remark 13. Theorem 6 shows that Conjecture 1 holds in the case $n = 2$. Theorem 8 shows that Conjecture 1 holds in the case $n = 3, d = 2$, while Theorem 10 shows that the conjecture holds when $n = 3, d = 3, k \leq 40$.

We now look at the case when $k < d^{n-1}$. There are obvious relations between the generators of $(f_1, \dots, f_{n+1})^k$ of type $f_1 \cdot f_2^k = f_2 \cdot (f_1 f_2^{k-1})$. These are of degree $d(k+1)$. Now, if the algebra $\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_{n+1})^k$ is zero in degree $d(k+1)$, these relations do not show up. This happens if

$$S_{n,d,k} = \binom{d+n-1}{n-1} \binom{k+n}{n} - \left(\binom{k+n}{n} (n+1) - \binom{k+1+n}{n} \right) - \binom{d(k+1)+n-1}{n-1} \geq 0.$$

Conjecture 2. *Let $k < d^{n-1}$ and suppose that for $R_{d,k,n} = \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_{n+1})^k$ we have $S_{n,d,k} \geq 0$. Then the Hilbert series of $R_{d,k,n}$ equals the Hilbert series of $\mathbb{C}[x_1, \dots, x_n]/I$, where I is an ideal generated by $\binom{k+n}{n}$ general forms of degree dk .*

It is known that $\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_{n+1})$ has the same Hilbert series whether the f_i 's are general forms of degree d , or d 'th powers of general linear forms [8]. In the second case we can assume that $(f_1, \dots, f_{n+1}) = (x_1^d, \dots, x_n^d, (x_1 + \dots + x_n)^d)$. We will end this section by giving an explicit relation of degree d^n in $\mathbb{C}[x_1^d, \dots, x_n^d, (x_1 + \dots + x_n)^d]$.

Theorem 14. *In $T = \mathbb{C}[x_1, \dots, x_{n+1}]/(x_1^d, \dots, x_{n+1}^d)^{d^{n-1}}$ we have*

$$F = \prod (x_1 + \epsilon_1 x_2 + \dots + \epsilon_{n-1} x_n + \epsilon_n x_{n+1}) = 0,$$

where the ϵ_i 's vary over all d^n combinations of a d 'th root of unity. In particular, $F/(x_1 + x_2 + \dots + x_{n+1})$ is a form of degree $d^n - 1$ which is in the kernel of the map $\times(x_1 + x_2 + \dots + x_{n+1}): T \rightarrow T$.

Proof. Let $G_d \subseteq \mathbb{C}^*$ be the group of d 'th roots of unity. The form F is invariant under the action of G_d^{n+1} , acting with multiplication on the variables, so it has to be a polynomial in $x_1^d, x_2^d, \dots, x_{n+1}^d$, which means that $F = 0$ in T . \square

Let us now consider the form F as an element in $\mathbb{C}[x_1, \dots, x_{n+1}]$. When $n+1=2$ and $d=2$, $F = (x_1 + x_2)(x_1 - x_2) = x_1^2 - x_2^2$ — the conjugate rule, and for general d , $F = x_1^d - (-1)^d x_2^d$.

When $n+1 \geq 3$, the form F is symmetric. To show this, notice first the the form is invariant under permutation of the variables x_2, x_3, \dots, x_{n+1} . Thus it is enough to show that F is invariant with respect to the transposition (12). Let G denote the result after letting (12) act on F . Let ϵ be a primitive d 'th root of unity and let $f_i = \prod (x_1 + \epsilon^i x_2 + \epsilon_2 x_3 + \dots + \epsilon_{n-1} x_n + \epsilon_n x_{n+1})$, so that $F = \prod_i f_i$. Let $g_i = \prod (\epsilon_1^i x_1 + x_2 + \epsilon_2 x_3 + \dots + \epsilon_{n-1} x_n + \epsilon_n x_{n+1})$. Since $n+1 \geq 3$, we have $f_i = \prod \epsilon^{d-i} (x_1 + \epsilon^i x_2 + \epsilon_2 x_3 + \dots + \epsilon_{n-1} x_n + \epsilon_n x_{n+1}) = g_{d-i}$, so $F = \prod_i f_i = \prod_i g_i = G$.

Example 15. In general, it seems hard to get an explicit description of F . We can get it for $d=2$, $n+1=3, 4, 5$ and for $n+1=d=3$. It is the symmetrization of the following polynomials.

$(n+1, d)$	
(3, 2)	$x^4 + x^2 y^2$
(4, 2)	$x^8 - 4x^6 y^2 + 6x^4 y^4 + 4x^4 y^2 z^2$
(5, 2)	$x^{16} - 8x^{14} y^2 + 28x^{12} y^4 + 40x^{12} y^2 z^2 - 56x^{10} y^6 - 72x^{10} y^4 z^2$ $-176x^{10} y^2 z^2 u^2 + 70x^8 y^8 + 40x^8 y^6 z^2 + 36x^8 y^4 z^4 + 344x^8 y^4 z^2 u^2$ $-757x^8 y^2 z^2 u^2 v^2 + 16x^6 y^6 z^4 - 416x^6 y^6 z^2 u^2 - 272x^6 y^4 z^4 u^2$ $+928x^6 y^4 z^2 u^2 v^2 + 2008x^4 y^4 z^4 u^4 - 1520x^4 y^4 z^4 u^2 v^2$
(3, 3)	$x^{27} + 36x^{24} y^3 - 9x^{21} y^3 z^3 + 684x^{18} y^9 - 234x^{18} y^6 z^3 + 3339x^{18} y^3 z^3 u^3$ $+126x^{15} y^{12} - 711x^{15} y^9 z^3 + 513x^{15} y^6 z^6 + 1512x^{15} y^6 z^3 u^3 - 990x^{12} y^{12} z^3$ $+2961x^{12} y^9 z^6 - 12222x^{12} y^6 z^6 u^3 + 278371x^{12} y^6 z^6 u^3 - 12171x^9 y^9 z^9$ $-6867x^9 y^9 z^6 u^3 + 120312x^9 y^6 z^6 u^6$

3. THE WEAK LEFSCHETZ PROPERTY

We now turn to the WLP for $T_{n,d,k} := \mathbb{C}[x_1, \dots, x_n]/(x_1^d, \dots, x_n^d)^k$. The algebra $T_{2,d,k}$ has the WLP, since quotients of $\mathbb{C}[x_1, x_2]$ has the Strong Lefschetz property ([6]), which implies the WLP. When $k=1$, the algebra is of the form $\mathbb{C}[x_1, \dots, x_n]/(x_1^d, \dots, x_n^d)$, which is a monomial complete intersection, and therefore also $T_{n,d,1}$ has the Strong Lefschetz property ([8]). When $d=1$, we have $T_{n,1,k} = \mathbb{C}[x_1, \dots, x_n]/(x_1, \dots, x_n)^k$. Since $\mathbb{C}[x_1, \dots, x_n]$ has the WLP, so has $\mathbb{C}[x_1, \dots, x_n]/(x_1, \dots, x_n)^k$. When $n \geq 3$ and $d, k \geq 2$, the situation is more involved.

The WLP for algebras given by monomial ideals have been studied before, for example in [3] and [2] but not for our kind of monomial ideal.

Since $T_{n,d,k}$ is a monomial algebra, $(\mathbb{C}^*)^n$ acts on $T_{n,d,k}$ and therefore any general linear form can be identified with $L = x_1 + \dots + x_n$. When taking the quotient by the ideal (L) , we get $\tilde{T}_{n,d,k} := \mathbb{C}[x_1, \dots, x_{n-1}]/(x_1^d, \dots, x_{n-1}^d, (x_1 + \dots + x_{n-1})^d)^k$. Thus $T_{n,d,k}$ has the WLP if and only if $[(1-t)T_{n,d,k}(t)] = \tilde{T}_{n,d,k}(t)$.

From Lemma 12 and Theorem 14 we can immediately get a negative result on the WLP.

Theorem 16. *Suppose that $k \geq d^{n-2}$, $n \geq 3$, $(n, d) \neq (3, 2)$. Then $T_{n,d,k}$ fails the WLP.*

Proof. By Lemma 12, $\tilde{T}_{n,d,k}$ is non-zero in degree dk . Thus it is enough to show that the map $\times(x_1 + x_2 + \cdots + x_n): (T_{n,d,k})_{dk-1} \rightarrow (T_{n,d,k})_{dk}$ is not injective.

Suppose that $k = d^{n-2}$. By Theorem 14, the form $(\prod(x_1 + \epsilon_1 x_2 + \cdots + \epsilon_{n-1} x_n))/(x_1 + x_2 + \cdots + x_n)$ of degree $dk - 1$ is in the kernel of the map. Hence it cannot be injective.

Suppose instead that $k > d^{n-2}$. Then $x_1^{k-d^{n-2}}(F/(x_1 + x_2 + \cdots + x_n))$ is in the kernel. \square

3.1. The case $n = 3$. We have a conjecture on the WLP for $T_{3,d,k}$.

Conjecture 3. *The algebra $\mathbb{C}[x, y, z]/(x^d, y^d, z^d)^k$ has the WLP if and only if one of the following conditions is satisfied.*

- (1) $d \leq 2$,
- (2) $k \leq 2$,
- (3) $d > k = 2j + 1 \in \{3, 7\}$, $d \neq (j + 2)(2\ell + 1)$,
- (4) $d > k = 2j + 1 > 2$, $k \notin \{3, 7\}$, $d \notin \{(j + 2)(2\ell + 1) - 1, (j + 2)(2\ell + 1), (j + 2)(2\ell + 1) + 1\}$,
- (5) $d > k = 2j > 2$, $d \notin \{(j + 1)(2\ell + 1) + \ell, (j + 1)(2\ell + 1) + \ell + 1\}$.

We have already proven that the algebra $T_{3,d,k}$ fails the WLP when $3 \leq d \leq k$ and that $T_{3,d,1}$ and $T_{3,1,k}$ have the WLP. We will now prove Conjecture 3 in some more cases.

Theorem 17. *The algebra $T_{3,2,k}$ has the WLP.*

Proof. It is enough to show that $\tilde{T}_{3,2,k}$ is zero in degree $2k$, which follows from the observation that $(x^2, y^2, (x + y)^2)^k = (x^2, y^2, xy)^k$. \square

In Theorem 18 below we prove the necessity of conditions (3)-(5) of Conjecture 3.

Theorem 18. *The algebra $T_{3,d,k}$ fails to have the WLP in the cases*

- (1) $k = 2j + 1 > 2$, $d = (j + 2)(2\ell + 1)$, where $\ell \geq 1$.
- (2) $k = 2j + 1 > 2$, $k \notin \{3, 7\}$, $d = (j + 2)(2\ell + 1) \pm 1$, where $\ell \geq 1$.
- (3) $k = 2j > 2$, $d \in \{(j + 1)(2\ell + 1) + \ell, (j + 1)(2\ell + 1) + \ell + 1\}$.

Proof. By [5] we have that the Hilbert series of $T_{3,d,k}$ is given by

$$T_{3,d,k}(t) = \frac{1 - \binom{k+2}{2}t^{dk} + (k^2 + 2k)t^{d(k+1)} - \binom{k+1}{2}t^{d(k+2)}}{(1-t)^3}.$$

From this, we get that in the range $dk \leq i \leq d(k + 1)$ the Hilbert function is given

$$\binom{i+2}{2} - \binom{i-dk+2}{2} \binom{k+2}{2}$$

with first difference equal to $(i + 1) - (i - dk + 1)(k + 1)(k + 2)/2$ which is positive when $i + 1 < d(k + 1)(k + 2)/(k + 3)$ and negative when $i + 1 > d(k + 1)(k + 2)/(k + 3)$. The turning point is in the interval where this expression for the Hilbert function is valid.

In order to show that the WLP fails, we will use the representation theory of the symmetric group, S_3 . Since the algebra is monomial, it is sufficient to consider multiplication by the linear form $L = x + y + z$ which is symmetric and hence gives an equivariant map. In all of the cases we consider, we have that $d = (2\ell + 1)(k + 3)/2 + \epsilon$, where $-1 \leq \epsilon \leq 1$. We will look at the multiplicity of the trivial and the alternating representations in $T_{3,d,k}$ in degrees $dk + 2\ell - 1$ and $dk + 2\ell$. We compute these multiplicities by means of the characters. In degree i of $T_{3,d,k}$ we have $\binom{i+2}{2} - \binom{k+2}{2} \binom{i-dk+2}{2}$ monomials that are all fixed by the identity permutation. The other two even permutations fixes only powers of xyz so the character is 1 if $i \equiv 0 \pmod{3}$ and $k \not\equiv 0 \pmod{3}$ and 0 otherwise. The transposition (12) fixes the monomials of the form $(xy)^m z^n$ and we have that the value of the character on the three transpositions is

$$\left\lfloor \frac{i+2}{2} \right\rfloor - \left\lfloor \frac{k+2}{2} \right\rfloor \left\lfloor \frac{i-dk+2}{2} \right\rfloor.$$

Thus we get that the multiplicity of the trivial representation is

$$\frac{1}{6} \left(\binom{i+2}{2} - \binom{k+2}{2} \binom{i-dk+2}{2} + m + 3 \left(\left\lfloor \frac{i+2}{2} \right\rfloor - \left\lfloor \frac{k+2}{2} \right\rfloor \left\lfloor \frac{i-dk+2}{2} \right\rfloor \right) \right)$$

where the middle term m is -2 , 0 or 2 depending on the congruence modulo 3 of the sum of the other terms. In the same way we get the multiplicity of the alternating representation as

$$\frac{1}{6} \left(\binom{i+2}{2} - \binom{k+2}{2} \binom{i-dk+2}{2} + m - 3 \left(\left\lfloor \frac{i+2}{2} \right\rfloor - \left\lfloor \frac{k+2}{2} \right\rfloor \left\lfloor \frac{i-dk+2}{2} \right\rfloor \right) \right)$$

When we compute the difference of the Hilbert function between degree $dk + 2\ell - 1$ and $dk + 2\ell$ we get

$$\begin{aligned} & \binom{dk+2\ell+2}{2} - \binom{dk+2\ell+1}{2} - \binom{k+2}{2} \left(\binom{2\ell+2}{2} - \binom{2\ell+1}{2} \right) \\ &= dk + 2\ell + 1 - \binom{k+2}{2} (2\ell + 1) = dk - (2\ell + 1) \frac{k(k+3)}{2} = k\epsilon. \end{aligned}$$

The difference in the number of monomials fixed by a transposition is

$$\begin{aligned} & \left\lfloor \frac{dk+2\ell+2}{2} \right\rfloor - \left\lfloor \frac{dk+2\ell+1}{2} \right\rfloor - \left\lfloor \frac{k+2}{2} \right\rfloor \left(\left\lfloor \frac{2\ell+2}{2} \right\rfloor - \left\lfloor \frac{2\ell+1}{2} \right\rfloor \right) \\ &= \left\lfloor \frac{dk}{2} \right\rfloor - \left\lfloor \frac{dk-1}{2} \right\rfloor - \left\lfloor \frac{k+2}{2} \right\rfloor = \begin{cases} -j-1 & \text{if } k \text{ and } d \text{ are odd,} \\ -j & \text{otherwise.} \end{cases} \end{aligned}$$

When k is odd and $\epsilon = \pm 1$, d is odd when j is even and the difference above can be expressed as $-2\lfloor j/2 \rfloor - 1$. When k is odd and $\epsilon = 0$, d is odd when j is odd and the same difference can be expressed as $-2\lfloor (j+1)/2 \rfloor$.

First we consider the case $k = 2j$ even and $\epsilon = \pm 1/2$. We compute the difference in the multiplicity of the trivial representation when $\epsilon > 0$ and the alternating representation when $\epsilon < 0$. This difference equals

$$\frac{1}{6} (\pm j + m \pm 3(-j)) = \mp \left\lfloor \frac{j+1}{3} \right\rfloor$$

which has opposite sign to the difference in the Hilbert function when $j + 1 \geq 3$. Thus the multiplication by $L = x + y + z$ cannot have maximal rank by Schur's lemma.

For the case $k = 2j + 1$ and $\epsilon = \pm 1$, we have the corresponding difference

$$\frac{1}{6} \left(\pm(2j + 1) + m \pm 3 \left(-2 \left\lfloor \frac{j}{2} \right\rfloor - 1 \right) \right) = \mp \left(\left\lfloor \frac{j}{2} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor \right).$$

Again, this has different sign than the difference in the Hilbert function when $\lfloor j/2 \rfloor > \lfloor j/3 \rfloor$, i.e., when $j = 2$ or $j \geq 4$.

In the last case $k = 2k + 1$ and $\epsilon = 0$ it is sufficient to show that the multiplicity of the trivial representation changes since the Hilbert function has difference zero. The difference in the multiplicity of the trivial representation is

$$\frac{1}{6} \left(0 + m + 3 \left(-2 \left\lfloor \frac{j+1}{2} \right\rfloor \right) \right) = - \left\lfloor \frac{j+1}{2} \right\rfloor$$

which is negative for all $j \geq 1$. □

Remark 19. We can see that the argument of the proof does not work when $\epsilon = \pm 3/2$ or $\epsilon = \pm 2$, so in these cases we have the same turning point for the Hilbert function of the three isotypic components.

Remark 20. By computations in Macaulay2, we have verified the unproven parts of Conjecture 3 for $d, k \leq 30$.

3.2. The general case. As the number of variables increases, the number of non-trivial pairs (d, k) for which $\mathbb{C}[x_1, \dots, x_n]/(x_1^d, \dots, x_n^d)^k$ has the WLP seems to decrease.

n	WLP
4	$(2, 2), (2, 3), (3, 2), (4, 2)$
5	$(2, 2), \dots, (2, 7), (3, 2), (4, 2), (4, 3), (6, 2)$
6	$(2, 2), (2, 3), (3, 2)$
7	$(2, 2), (2, 3), (3, 2)$
8	$(2, 2), (3, 2)$
9	$(2, 2), (2, 3), (3, 2)$
10	$(2, 2), (3, 2)$
11	$(2, 2), (2, 3)$
12, 14, 16	$(2, 2)$
$2a + 1$	$(2, 2)$

TABLE 1. The right hand column consists of pairs (d, k) with $d, k \geq 2$ for which $\mathbb{C}[x_1, \dots, x_n]/(x_1^d, \dots, x_n^d)^k$ has the WLP, detected by computations in Macaulay2, except for the last row, which relies on Theorem 22.

In Table 1 we list the non-trivial cases where have been able detect the WLP. Based upon these observations together with some negative results for the WLP in positive characteristic, we believe that Table 1 is complete, except for the trivial pairs, when $n \leq 10$. When $n \geq 11$, we guess that the WLP holds at most in the cases $(2, 2), (2, 3), (3, 2)$

for n odd, and at most in the cases $(2, 2), (3, 2)$ for n even. We can support our guesses with two theoretical results.

Theorem 21. • *The algebra $T_{4,2,k}$ has the WLP if and only if $k \leq 3$.*

- *The algebra $T_{4,3,k}$ has the WLP if and only if $k \leq 2$.*
- *The algebra $T_{4,4,k}$ has the WLP if and only if $k \leq 2$.*
- *The algebra $T_{5,2,k}$ has the WLP if and only if $k \leq 7$.*

Proof. By Theorem 16, the algebra $T_{4,2,k}$ fails the WLP when $k \geq 2^2 = 4$, $T_{4,3,k}$ fails the WLP when $k \geq 3^2 = 9$, $T_{4,4,k}$ fails the WLP when $k \geq 4^2 = 16$, $T_{5,2,k}$ fails the WLP when $k \geq 2^3 = 8$,

Hence it is enough to check the WLP over \mathbb{Q} for the remaining cases. This has been done with Macaulay2. \square

Finally, we can prove that $\mathbb{C}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)^2$ has the WLP when n is odd. We believe that it is also true for n even, but our method of proof does not apply in that case.

Theorem 22. *For any odd n , the algebra $\mathbb{C}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)^2$ has the WLP.*

Proof. Let $R = \mathbb{C}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)^2$. When $d \geq 2$, write $R_d = A_{d,1} \oplus \dots \oplus \dots \oplus A_{d,n-1} \oplus B_d$, where $A_{d,i}$ is spanned by all monomials of the form $x_i^2 M$ with M a squarefree monomial, and where B_d is the set of squarefree monomials of degree d together with all monomials of the form $x_n^2 M$, with M squarefree. We have $|A_{d,i}| = \binom{n}{d-2}$ and $|B_d| = \binom{n}{d} + \binom{n}{d-2}$, where $|V|$ denotes the dimension of the vector space V . Notice that $R_i A_{d,j} \subset A_{i+d,j}$.

Multiplication by $(x_1 + \dots + x_n)$ on $A_{d,i}$ agrees with multiplication by $(x_1 + \dots + x_n)$ on the basis in degree $d-2$ in $\mathbb{C}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$. This algebra has the SLP, so multiplication by $(x_1 + \dots + x_n)$ on $A_{d,i}$ has full rank.

Since $(x_1^2, \dots, x_n^2)^2 \subseteq (x_1^2, \dots, x_{n-1}^2, x_n^4)$, the dimension of $(x_1 + \dots + x_n)B_d$ in R is greater than or equal to the dimension of $(x_1 + \dots + x_n)B_d$ in the algebra $C := \mathbb{C}[x_1, \dots, x_n]/(x_1^2, \dots, x_{n-1}^2, x_n^4)$, where we abuse notation and regard B_d as a part of C . This algebra also has the SLP, so the dimension of $(x_1 + \dots + x_n)B_d$ in C equals $\min(|B_d|, |B_{d+1}|) = \min\left(\binom{n}{d} + \binom{n}{d-2}, \binom{n}{d+1} + \binom{n}{d-1}\right)$.

If $|B_d| \leq |B_{d+1}|$ and $|A_d| \leq |A_{d+1}|$ we can conclude that multiplication by $(x_1 + \dots + x_n)$ on R_d is injective. If $|B_d| \geq |B_{d+1}|$ and $|A_{d,i}| \geq |A_{d+1,i}|$, we can conclude that multiplication by $(x_1 + \dots + x_n)$ on R_d is surjective.

Thus we are left with two cases.

In the first case we have $|B_d| < |B_{d+1}|$ and $|A_{d,i}| > |A_{d+1,i}|$, that is, we have the inequalities $\binom{n}{d} + \binom{n}{d-2} < \binom{n}{d+1} + \binom{n}{d-1}$ and $\binom{n}{d-2} > \binom{n}{d-1}$. But $\binom{n}{d-2} > \binom{n}{d-1}$ implies that $\binom{n}{d} > \binom{n}{d+1}$, so $\binom{n}{d} + \binom{n}{d-2} > \binom{n}{d+1} + \binom{n}{d-1}$. Thus the inequalities $|B_d| < |B_{d+1}|$ and $|A_{d,i}| > |A_{d+1,i}|$ cannot be simultaneously satisfied.

In the second case we have $|B_d| > |B_{d+1}|$ and $|A_{d,i}| < |A_{d+1,i}|$. For $\binom{n}{d-2} < \binom{n}{d-1}$ to hold we need to have $d-1 \leq \lfloor n/2 \rfloor$. For $\binom{n}{d} + \binom{n}{d-2} > \binom{n}{d+1} + \binom{n}{d-1}$ to simultaneously

hold, we especially need the inequality $\binom{n}{d} > \binom{n}{d+1}$ to hold. This inequality is satisfied if and only if $d \geq \lfloor (n+1)/2 \rfloor$.

Now the inequalities $d-1 \leq \lfloor n/2 \rfloor$ and $d \geq \lfloor (n+1)/2 \rfloor$ together gives us that $d = \lfloor n/2 \rfloor + 1$. In the case $n = 2k+1$, we have $|B_{k+1}| = \binom{2k+1}{k+1} = \binom{2k+1}{k+1} + \binom{2k+1}{k-1} = \binom{2k+1}{k} + \binom{2k+1}{k+2} = |B_{k+2}|$, so the inequality $|B_d| > |B_{d+1}|$ is not fulfilled. This shows that the case $|B_d| > |B_{d+1}|$ and $|A_{d,i}| < |A_{d+1,i}|$ is empty when n is odd, that is, R has the WLP when n is odd.

Remark: When $n = 2k$, we can however not draw any conclusion regarding the rank of the map $\times(x_1 + \cdots + x_n)$ from degree $k+1$ to $k+2$.

□

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